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$\mathcal{H}_\infty$ Control of Nonlinear Systems via Output Feedback: A Class of Controllers

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Abstract

The standard state space solutions to the $\mathcal{H}_\infty$ control problem for linear time invariant systems are generalized to nonlinear time-invariant systems. A class of nonlinear $\mathcal{H}_\infty$-controllers are parametrized as nonlinear fractional transformations on contractive, stable free nonlinear parameters. As in the linear case, the $\mathcal{H}_\infty$ control problem is solved by its reduction to state feedback and output injection problems, together with a separation argument. The sufficient conditions for $\mathcal{H}_\infty$-control problem to be solved are also derived with this machinery. The solvability for nonlinear $\mathcal{H}_\infty$-control problem requires positive definite solutions to two parallel decoupled Hamilton-Jacobi inequalities and these two solutions satisfy an additional coupling condition. An illustrative example, which deals with a passive plant, is given at the end. This paper is a condensed version of [11].

1 Introduction

Linear $\mathcal{H}_\infty$ control theory has been a very popular research area since it was originally formulated by Zames (cf. [2, 5, 4]). The simplicity of the characterization of state space $\mathcal{H}_\infty$-control theory together with its clear connections with traditional methods in optimal control [4] have stimulated several attempts to generalize the linear $\mathcal{H}_\infty$ results in state space to nonlinear systems [15, 8, 1, 11]. We will use the accepted but unfortunate misnomer "nonlinear $\mathcal{H}_\infty$" to describe this research direction, which will be pursued further in this paper.

Basically, in the nonlinear generalizations, the necessary or sufficient conditions for the the $\mathcal{H}_\infty$-control problem to be solvable are characterized in terms of some Hamilton-Jacobi equations or inequalities [11, 10, 15, 8, 1, 6, 16]. In [15], van der Schaft showed that a sufficient condition for the state feedback $\mathcal{H}_\infty$-control problem to be solvable is that the corresponding HJI has a positive solution; Isidori and Astolfi showed that the solution to the output feedback $\mathcal{H}_\infty$-control problem requires the existence of positive definite solutions of two coupled HJIs [8, 6]. Ball, Helton and Walker derived some necessary conditions for a stronger output feedback $\mathcal{H}_\infty$-control problem to be solvable [1]; these conditions are that two HJIs have positive solutions and the solutions are coupled locally; they confirmed the separation principle for the nonlinear $\mathcal{H}_\infty$-control system. Van der Schaft and Isidori also considered the same necessity aspect and derived similar results [16, 7]. Some other extensions and alternative approaches for the $\mathcal{H}_\infty$-control problems are reported in [16, 10, 12].

Our goal in this paper is to systematically examine the nonlinear output feedback $\mathcal{H}_\infty$-control problem in state space and obtain an $\mathcal{H}_\infty$ controller parametrization. Both plant and controllers are nonlinear time-invariant and realized as (control) input-affine state-space equations. We deal with this $\mathcal{H}_\infty$-control problem using similar techniques in the linear case [4]. The $\mathcal{H}_\infty$ control problem is solved by its reduction to state feedback and output injection problems, together with a separation argument. Sufficient conditions for the output feedback $\mathcal{H}_\infty$-control problem to be locally solvable are also derived using this machinery. The solvability of the $\mathcal{H}_\infty$-control problem requires the positive definite solutions to two parallel decoupled HJIs and these two solutions satisfy an additional condition. A class of $\mathcal{H}_\infty$-controllers are parametrized as a nonlinear fractional transformation on contractive, stable free nonlinear operators. In each case, the stability of the resulting closed loop system is confirmed via the use of the stability theorem of hierarchical systems [17]. Any concept or result in this paper is local unless otherwise noted.

This paper is organized as follows: In section 2, some background material related to the $\mathcal{L}_2$-gains is given, the $\mathcal{H}_\infty$-control problem is stated. In section 3, some tools are provided, the $\mathcal{H}_\infty$-control problem for some systems with special structures are considered; the solvability conditions or controller parametrizations are given. In section 4, the main results of this paper, solutions to the output feedback $\mathcal{H}_\infty$-control problem, are given; the solvability of this problem requires the coupled positive definite solutions to two decoupled HJIs; a class of $\mathcal{H}_\infty$-controllers are parametrized. As an illustrative example, the $\mathcal{H}_\infty$ control design for a passive system is conducted.

The following conventions are made in this paper. $\mathbf{R}$ is the set of real numbers, $\mathbf{R}^+ := [0,\infty) \subset \mathbf{R}$. $\mathbf{R}^n$ is n-dimensional real Euclidean space; if $u \in \mathbf{R}^n$, then $\|u\|$ is Euclidean norm of u. $\mathbf{R}^{n\times m}$ is the set of real $n \times m$ matrices; if $A \in \mathbf{R}^{n\times m}$, then $A^T \in \mathbf{R}^{m\times n}$ is the transpose of A. By $P > 0$ ($P \geq 0$) for some matrix $P = P^T \in \mathbf{R}^{n\times n}$ we mean that the matrix is (semi-)positive definite. A function is said to be of class $\mathbf{C}^k$ if it is continuously differentiable k times; so $\mathbf{C}^0$ stands for the class of continuous functions. $\|\cdot\|$ stands for the **Euclidean norm.** $\mathcal{B}_r := \{x \in \mathbf{R}^n \mid \|x\| < r, \text{for some integer } n > 0\}$; we shall not specify the dimension of the environmental space, and always use the same r to denote its radius without confusion. $\mathcal{L}_2[0,T], \mathcal{L}_2[0,\infty)$ are two standard Lebesgue Spaces; $\mathcal{L}_2^e[0,\infty)$ is the extended space of $\mathcal{L}_2[0,\infty)$. $\Omega(G,K)$ represents **fractional transformation** of operator G on operator K; $\Sigma(M_1, M_2)$ stands for the **Redheffer product** of operators M_1 and M_2 (see [13, 11] for exact definitions).

2 $\mathcal{L}_2$-Gains and Nonlinear $\mathcal{H}_\infty$-Control Problems

In this section, some background material about $\mathcal{L}_2$-gain analysis of nonlinear systems is provided. The reader is referred to [18, 15, 11] for more details.

2.1 $\mathcal{L}_2$-Gains of Nonlinear Systems

Consider the following input-affine nonlinear time-invariant (NLTI) system:

$$G : \begin{cases} \dot{x} = f(x) + g(x)w \\ z = h(x) + k(x)w \end{cases}$$

Where $x \in \mathbf{R}^n$ is state vector, $w \in \mathbf{R}^p$ and $z \in \mathbf{R}^q$ are input and output vectors, respectively. We will assume $f, g, h, k \in \mathbf{C}^0$, and $f(0) = 0, h(0) = 0$. Therefore, $0 \in \mathbf{R}^n$ is the equilibrium of the system with $w = 0$. The state transition function $\phi : \mathbf{R}^+ \times \mathbf{R}^n \times \mathbf{R}^p \to \mathbf{R}^n$ is so defined that $x = \phi(T, x_0, w^*)$ means that system G evolves from initial state x_0 to state x in time T under the control action w^*.

Definition 2.1 *(i) A system G (or $[f(x), g(x)]$) is* **reachable** *from 0 if for all $x \in \mathbb{R}^n$, there exist $T \in R^+$ and $w^*(t) \in \mathcal{L}_2[0,T]$ such that $x = \phi(T, 0, w^*)$;*

(ii) A system G (or $[h(x), f(x)]$) is (zero-state) **detectable** *if for all $x \in \mathbb{R}^n$, $h(\phi(t,x,0)) = 0 \Rightarrow \phi(t,x,0) \rightarrow 0$ as $t \rightarrow \infty$; it is (zero-state)* **observable** *if for all $x \in \mathbb{R}^n$, $h(\phi(t,x,0)) = 0 \Rightarrow \phi(t,x,0) = 0$ for all $t \in \mathbb{R}^+$.*

Definition 2.2 *A system G is said to have $\mathcal{L}_2$-gain less than or equal to γ for some $\gamma > 0$ if*

$$\int_0^T \|z(t)\|^2\, dt \leq \gamma^2 \int_0^T \|w(t)\|^2\, dt$$

for all $T \geq 0$ and $w(t) \in \mathcal{L}_2[0,T]$, and $z(t) = h(\phi(t,0,w(t))) + k(\phi(t,0,w(t)))w(t)$.

Note that in the above definition, we take the initial state $x(0) = 0$. In the following discussion, we only consider the case $\gamma = 1$ without loss of generality. It has been shown [18] that the system has $\mathcal{L}_2$-gain ≤ 1 if and only if there exists a positive $V : \mathbb{R}^n \rightarrow \mathbb{R}^+$ with $V(0) = 0$ such that the following inequality holds,

$$V(x) - V(x_0) \leq \int_0^T (\|w(t)\|^2 - \|z(t)\|^2) dt \qquad (2.1)$$

where $x = \phi(T, x_0, w(t))$ and $w(t) \in \mathcal{L}_2[0,T]$, and

$$V_a(x) := \sup_{w \in \mathcal{L}_2[0,\infty), x(0)=x} - \int_0^\infty (\|w(t)\|^2 - \|z(t)\|^2) dt. \qquad (2.2)$$

is well defined; moreover, $V_a(\cdot)$ is also a solution to (2.1), and the solutions to (2.1) form a convex set, and any solution $V(x) \geq 0$ for $x \in \mathbb{R}^n$ with $V(0) = 0$ satisfies $V(x) \geq V_a(x)$. If $V : \mathbb{R}^n \rightarrow \mathbb{R}$ is of class $\mathbf{C}^1$, define

$$\mathcal{H}(V,x) := \frac{\partial V}{\partial x}(x)(f(x) - g(x)R^{-1}(x)k^T(x)h(x)) +$$

$$\frac{1}{4}\frac{\partial V}{\partial x}(x)g(x)R^{-1}(x)g^T(x)\frac{\partial V^T}{\partial x}(x) + h^T(x)(I - k(x)k^T(x))^{-1}h(x) \qquad (2.3)$$

We have the following result, which is pretty standard and whose proof is given in [15, 11].

Theorem 2.1 *Consider G with $R(x) := I - k^T(x)k(x) > 0$ for all $x \in \mathbb{R}^n$, V_a and V are defined by (2.2) and (2.1).*

i) G has $\mathcal{L}_2$-gain ≤ 1 and $V_a(x)$ with $V_a(0) = 0$ is of class $\mathbf{C}^1$ if and only if the Hamilton-Jacobi equation (HJE) *$\mathcal{H}(V_a, x) = 0$ holds;*

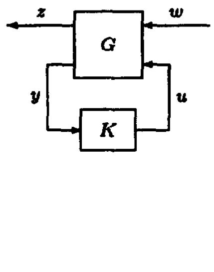
ii) G has $\mathcal{L}_2$-gain ≤ 1 and $V(x)$ with $V(0) = 0$ is of class $\mathbf{C}^1$ if and only if the Hamilton-Jacobi inequality (HJI) *$\mathcal{H}(V,x) \leq 0$ holds.*

Recall that $V : \mathbb{R}^n \rightarrow \mathbb{R}^+$ is locally positive-definite if there exists $r > 0$ such that for $x \in \mathcal{B}_r$, $V(x) = 0 \Rightarrow x = 0$; it is globally positive-definite if $V(x) = 0 \Rightarrow x = 0$, and $\lim_{x \rightarrow \infty} V(x) = \infty$.

To close this subsection, we denote $\mathcal{FG}$ as the class of all input-affine NLTI systems which are asymptotically stable with zero input and related HJI has a positive definite solution.

2.2 $\mathcal{H}_\infty$-Control Problem Statement

The feedback configuration for the $\mathcal{H}_\infty$-control synthesis problem is depicted as follows



where G is the nonlinear plant with two sets of inputs: the exogenous disturbance inputs w and the control inputs u, and two sets of outputs: the measured outputs y and the regulated outputs z. And K is the controller to be designed. Both G and K are nonlinear time-invariant and can be realized as control-affine state-space equations:

$$G : \begin{cases} \dot{x} &= f(x) + g_1(x)w + g_2(x)u \\ z &= h_1(x) + k_{11}(x)w + k_{12}(x)u \\ y &= h_2(x) + k_{21}(x)w + k_{22}(x)u \end{cases}$$

where $f, g_i, h_i, k_{ij} \in \mathbf{C}^2$ and $f(0) = 0, h_1(0) = 0, h_2(0) = 0$; x, w, u, z, and y are assumed to have dimensions n, p_1, p_2, q_1, and q_2, respectively.

$$K : \begin{cases} \dot{\hat{x}} &= a(\hat{x}) + b(\hat{x})y \\ u &= c(\hat{x}) + d(\hat{x})y \end{cases}$$

with $a, b, c, d \in \mathbf{C}^2$ and $a(0) = 0, c(0) = 0$.

The initial states for both plant and controller are $x(0) = 0$ and $\hat{x}(0) = 0$. The closed loop system can be denoted as nonlinear operator $\Omega(G, K)$ which is the fractional transformation of G on K. We shall consider the following output feedback (OF) $\mathcal{H}_\infty$-control problem.

$\mathcal{H}_\infty$-Control Problem: Find a output feedback controller K (or a class controllers) if any, such that the closed-loop system $\Omega(G,K)$ is asymptotically stable with $w = 0$ and has $\mathcal{L}_2$-gain ≤ 1, i.e.

$$\int_0^T (\|w(t)\|^2 - \|z(t)\|^2) dt \geq 0;$$

for all $T \in \mathbb{R}^+$.

The following assumptions on system structure are made:

[A1]: $k_{11}(x) = 0, k_{22}(x) = 0$;

[A2]: $k_{12}^T(x) \begin{bmatrix} h_1(x) & k_{12}(x) \end{bmatrix} = \begin{bmatrix} 0 & I \end{bmatrix}$;

[A3]: $\begin{bmatrix} g_1(x) \\ k_{21}(x) \end{bmatrix} k_{21}^T(x) = \begin{bmatrix} 0 \\ I \end{bmatrix}$;

It is noted that, if the $\mathcal{H}_\infty$-control problem is considered, then many nonlinear systems can be transformed into the systems with the above structural constraints as in the linear case [14, 4]).

3 Tools for $\mathcal{H}_\infty$-Control Synthesis

The OF $\mathcal{H}_\infty$-control problem is solved by its reduction into some simpler sub-problems: state feedback and output injection. In this section, we will consider these problems to develop the necessary tools.

3.1 Full Information Problem

The full information (FI) structure is as follows,

$$G_{FI} : \begin{cases} \dot{x} &= f(x) + g_1(x)w + g_2(x)u \\ z &= h_1(x) + k_{12}(x)u \\ y &= \begin{bmatrix} x \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} w \end{cases}$$

The assumptions relevant to FI problem are inherited from OF structure, i.e.,

[A2]: $k_{12}^T(x) \begin{bmatrix} h_1(x) & k_{12}(x) \end{bmatrix} = \begin{bmatrix} 0 & I \end{bmatrix}$;

[A4]: $[h_1(x), f(x)]$ is zero-state detectable.

The FI $\mathcal{H}_\infty$-control problem was first explicitly introduced and solved by Van der Schaft [15] (see also Isidori [6]). The solutions to $\mathcal{H}_\infty$-control problem are related to the following HJI:

$$\mathcal{H}_{FI}(V,x) := \frac{\partial V}{\partial x}(x)f(x) + h_1^T(x)h_1(x)$$

$$+ \frac{1}{4}\frac{\partial V}{\partial x}(x)(g_1(x)g_1^T(x) - g_2(x)g_2^T(x))\frac{\partial V^T}{\partial x}(x) \leq 0. \qquad (3.1)$$

The following theorem reveals a property related to HJI (3.1).

Theorem 3.1 *$\mathcal{H}_{FI}(V,x) \le 0$ has a $\mathbf{C}^3$ solution $V(x)$ with $V(0)=0$ if and only if there is $F(x)$ such that*

$$\frac{\partial V}{\partial x}(x)(f(x)+g_2(x)F(x)) + \frac{1}{4}\frac{\partial V}{\partial x}(x)g_1(x)g_1^T(x)\frac{\partial V^T}{\partial x}(x) +$$

$$+ (h_1(x)+k_{12}(x)F(x))^T(h_1(x)+k_{12}(x)F(x)) \le 0. \qquad (3.2)$$

Moreover, if $V(x)$ satisfies $\mathcal{H}_{FI}(V,x) \le 0$ with $V(0)=0$, then $F(x)$ can be taken as $F_0(x) := -\frac{1}{2}g_2^T(x)\frac{\partial V^T}{\partial x}(x)$.

A proof of the above theorem is given in [11, 15]. A solution to FI $\mathcal{H}_\infty$-control problem in terms of state-feedback are restated as following theorem (see also [15, 6, 8]).

Theorem 3.2 *Consider G_{FI}. Suppose there exists $\mathbf{C}^3$ positive definite function $V(x) \ge 0$ such that $\mathcal{H}_{FI}(V,x) \le 0$, with $V(0)=0$. Then the $\mathcal{H}_\infty$-control problem for FI is solvable. Moreover, such a state feedback FI $\mathcal{H}_\infty$-controller is given by $u = -\frac{1}{2}g_2^T(x)\frac{\partial V^T}{\partial x}(x)$.*

The reader is referred to [8] for one treatment of full information $\mathcal{H}_\infty$-controller parametrization.

3.2 Full Control Problem

The full control (FC) structure is defined as

$$G_{FC}: \left\{ \begin{array}{ccccccc} \dot{x} & = & f(x) & + & g_1(x)w & + & \begin{bmatrix} I & 0 \end{bmatrix} u \\ z & = & h_1(x) & + & & & \begin{bmatrix} 0 & I \end{bmatrix} u \\ y & = & h_2(x) & + & k_{21}(x)w & & \end{array} \right.$$

The assumptions are as follows,

[A3]: $\begin{bmatrix} g_1(x) \\ k_{21}(x) \end{bmatrix} k_{21}^T(x) = \begin{bmatrix} 0 \\ I \end{bmatrix}$;

[A5]: $[h_2(x), f(x)]$ is zero-state detectable.

The solvability of FC $\mathcal{H}_\infty$-control problem is also related to a HJI:

$$\mathcal{H}_{FC}(U,x) := \frac{\partial U}{\partial x}(x)f(x) + \frac{1}{4}\frac{\partial U}{\partial x}(x)g_1(x)g_1^T(x)\frac{\partial U^T}{\partial x}(x) +$$

$$+ h_1^T(x)h_1(x) - h_2^T(x)h_2(x) \le 0. \qquad (3.3)$$

Theorem 3.3 *If a $\mathbf{C}^3$ function $U(x)$ with $U(0)=0$ satisfies*

$$\mathcal{H}_{OI}(U,L,x) := \frac{\partial U}{\partial x}(x)(f(x)+L(x)h_2(x)) + h_1^T(x)h_1(x) +$$

$$+\frac{1}{4}\frac{\partial U}{\partial x}(x)(g_1(x)+L(x)k_{21}(x))(g_1(x)+L(x)k_{21}(x))^T\frac{\partial U^T}{\partial x}(x) \le 0 \qquad (3.4)$$

for some $\mathbf{C}^2$ $L(x)$, then $U(x)$ satisfies $\mathcal{H}_{FC}(U,x) \le 0$ with $U(0) = 0$ as well.

Conversely, if a $\mathbf{C}^3$ $U(x)$ satisfies $\mathcal{H}_{FC}(U,x) \le 0$ with $U(0) = 0$, and a $\mathbf{C}^2$ $L_0(x)$ is such that $\frac{\partial U}{\partial x}(x)L_0(x) = -2h_2^T(x)$, then $\mathcal{H}_{OI}(U,L_0,x) \le 0$.

The proof of the preceding theorem is given in [11]. The following theory provides a output injection solution to the FC $\mathcal{H}_\infty$-control problem.

Theorem 3.4 *Suppose there exists a $\mathbf{C}^3$ positive definite function $U(x)$ such that $\mathcal{H}_{FC}(U,x) \le 0$. If $\frac{\partial U}{\partial x}(x)L_0(x) = -2h_2^T(x)$ holds for some $\mathbf{C}^2$ $L_0(x)$, then the FC $\mathcal{H}_\infty$-control problem is solvable. Moreover, such a controller is given by "output injection" $u(x) = \begin{bmatrix} L_0(x) \\ 0 \end{bmatrix} y$.*

Proof

We first prove $\mathcal{L}_2$-**gain** ≤ 1.

$$\dot{U}(x) = \frac{\partial U}{\partial x}(x)(f(x)+g_1(x)w + \begin{bmatrix} I & 0 \end{bmatrix} u)$$

$$\le -\frac{1}{4}\frac{\partial U}{\partial x}(x)g_1(x)g_1^T(x)\frac{\partial U^T}{\partial x}(x) - h_1^T(x)h_1(x) + h_2^T(x)h_2(x) +$$

$$+\frac{\partial U}{\partial x}(x)g_1(x)w + \begin{bmatrix} \frac{\partial U}{\partial x}(x) & 0 \end{bmatrix} u$$

$$= \left\| \begin{bmatrix} 0 & I \end{bmatrix} u \right\|^2 - \left\| w - g_1^T(x)\frac{\partial U^T}{\partial x}(x) + k_{21}^T(x)h_2(x) \right\|^2 +$$

$$- \|z\|^2 + \|w\|^2 + (\begin{bmatrix} \frac{\partial U}{\partial x}(x) & 2h_1^T(x) \end{bmatrix} u + 2h_2^T(x)(h_2(x)+k_{21}(x)w))$$

By assumption **[A3]**, $k_{21}(x)k_{21}^T(x) = I$ for all $x \in \mathbf{R}^n$, and $y = h_2(x) + k_{21}(x)w = k_{21}(x)(w - g_1^T(x)\frac{\partial U^T}{\partial x}(x) + k_{21}^T(x)h_2(x))$, thus,

$$\|y\|^2 \le \left\| w - g_1^T(x)\frac{\partial U^T}{\partial x}(x) + k_{21}^T(x)h_2(x) \right\|^2$$

Therefore,

$$\dot{U}(x) \le \left\| \begin{bmatrix} 0 & I \end{bmatrix} u \right\|^2 - \|y\|^2 - \|z\|^2 + \|w\|^2 +$$

$$+ (\begin{bmatrix} \frac{\partial U}{\partial x}(x) & 2h_1^T(x) \end{bmatrix} u + 2h_2^T(x)y). \qquad (3.5)$$

Note that

$$\dot{U}(x) \le -\|y\|^2 - \|z\|^2 + \|w\|^2 \le -\|z\|^2 + \|w\|^2, \qquad (3.6)$$

if u is such that $\begin{bmatrix} 0 & I \end{bmatrix} u = 0$ and $\begin{bmatrix} \frac{\partial U}{\partial x}(x) & 2h_1^T(x) \end{bmatrix} u + 2h_2^T(x)y = 0$, but it is guaranteed by taking the controller as the given "output injection" $u(x) = \begin{bmatrix} L_0(x) \\ 0 \end{bmatrix} y$, where $L_0(x)$ is such that $\frac{\partial U}{\partial x}(x)L_0(x) = -2h_2^T(x)$. It follows that

$$\int_0^T (\|w\|^2 - \|z\|^2)dt \ge U(x(T)) - U(0) = U(x(T)) \ge 0$$

for all $T \ge 0$. This confirmed the $\mathcal{L}_2$-gain ≤ 1 by the given output injection.

Next, consider the **stability**. Since $U(x)$ is positive definite by assumption, it can be used as a Lyapunov function. Let $w = 0$, (3.6) implies

$$\dot{U}(x) \le -\|z(t)\|^2 - \|h_2(x(t))\|^2 \le 0.$$

So $\dot{U}(x) = 0 \Rightarrow h_2(x(t)) = 0 \Rightarrow x(t) \to 0$ as $t \to \infty$ by assumption **[A5]**. LaSalle's Theorem implies $\dot{x} = f(x) + L_0(x)h_2(x)$ is asymptotically stable. □

Let $L_1(x)$ be such that $\frac{\partial U}{\partial x}(x)L_1(x) = -2h_1^T(x)$. Take $u_1 = L_0(x)y + L_1(x)u_2$, then

$$\frac{\partial U}{\partial x}(x)u_1 + 2h_1^T(x)u_2 + 2h_2^T(x)y = 0.$$

Therefore, (3.5) implies

$$\dot{U}(x) \le \|u_2\|^2 - \|y\|^2 - \|z\|^2 + \|w\|^2$$

Let $u_2 = Qy$ with $Q \in \mathcal{FG}$ (so $u_1 = L_0(x)y + L_2(x)Qy$ then), Q can be assumed to have the following realization

$$\left\{ \begin{array}{l} \dot{\xi} = a(\xi) + b(\xi)y \\ u_2 = c(\xi) + d(\xi)y \end{array} \right.$$

Then there exists $U_Q(\xi) \ge 0$ positive definite such that

$$\dot{U}_Q(\xi) \le \|y\|^2 - \|u_2\|^2$$

and $\dot{\xi} = a(\xi)$ is asymptotically stable.

Define $W(x,\xi) = U(x) + U_Q(\xi)$ for $(x,\xi) \in \mathbf{R}^n \times \mathbf{R}^n$, then $W(x,\xi) \ge 0$ is positive definite, and

$$\dot{W}(x,\xi) = \dot{U}(x) + \dot{U}_Q(\xi) \le \|w\|^2 - \|z\|^2 .$$

Therefore,

$$\int_0^T (\|w\|^2 - \|z\|^2)dt \ge W(x(T),\xi(T)) \ge 0.$$

for all $T \ge 0$.

Thus, we motivated the characterization of a class of controllers [11].

Theorem 3.5 *The assumptions are the same as in the last Theorem. If in addition, U is such that $\mathcal{H}_{FC}(U,x)$ is negative definite, and $L_1(x)$ also satisfies $\frac{\partial U}{\partial x}(x)L_1(x) = -2h_1^T(x)$, then*

$$u = \begin{bmatrix} L_0(x) + L_1(x)Q \\ Q \end{bmatrix} y$$

for all $Q \in \mathcal{FG}$ also solves the FC $\mathcal{H}_\infty$-control problem.

3.3 Output Estimation Problem

In this subsection, we consider a special OF $\mathcal{H}_\infty$-control problem to the following structure called output estimation (OE),

$$G_{OE} : \left\{ \begin{array}{lllllll} \dot{x} & = & f(x) & + & g_1(x)w & + & g_0(x)u \\ z & = & h_1(x) & + & & & u \\ y & = & h_2(x) & + & k_{21}(x)w & & \end{array} \right.$$

The assumptions for this structure are

[A3]: $\begin{bmatrix} g_1(x) \\ k_{21}(x) \end{bmatrix} k_{21}^T(x) = \begin{bmatrix} 0 \\ I \end{bmatrix}$;

[A5]: $[h_2(x), f(x)]$ is (locally) zero-state detectable.

Note that for linear system, OE and FC structure are equivalent in the sense that there is some system P_{OE} such that $G_{FC} = \Sigma(G_{OE}, P_{OE})$; therefore, the OE $\mathcal{H}_\infty$-control controllers can be constructed in terms of FC $\mathcal{H}_\infty$-control controllers: $K_{OE} = \Omega(P_{OE}, K_{FC})$, since $\Omega(G_{OE}, K_{OE}) = \Omega(G_{OE}, \Omega(P_{OE}, K_{FC})) = \Omega(\Sigma(G_{OE}, P_{OE}), K_{FC}) = \Omega(G_{FC}, K_{FC})$ (See [4, 9] for more discussion about this). In the following we use the similar idea to construct the OE $\mathcal{H}_\infty$-controllers. But the stability issue is not as trivial as in the linear case. The reader is referred to [11] for the asymptotic stability issue which is skipped here.

Theorem 3.6 *Consider G_{OE}, suppose there exists a $\mathbf{C}^3$ positive definite solution $U(x)$ to HJI: $\mathcal{H}_{FC}(U,x) \le 0$, with $U(0) = 0$; and $U(x)$ makes the Hessian matrix of $\mathcal{H}_{FC}(U,x)$ with respect to $x \in \mathbf{R}^n$ be negative definite at 0. If $\mathbf{C}^2$ $L_0(x)$ satisfies*

$$\frac{\partial U}{\partial x}(x)L_0(x) = -2h_2^T(x),$$

then there is a controller which makes the closed loop system have $\mathcal{L}_2$-gain ≤ 1; and such a controller is given by

$$K_{OE} : \left\{ \begin{array}{l} \dot{\tilde{x}} = f(\tilde{x}) - g_0(\tilde{x})h_1(\tilde{x}) + L_0(\tilde{x})h_2(\tilde{x}) - L_0(\tilde{x})y \\ u = -h_1(\tilde{x}) \end{array} \right.$$

Lemma 3.7 *Suppose the positive definite $U(x) \ge 0$ is such that $\mathcal{H}_{FC}(U,x)$ is negative definite. Let $x, \tilde{x}$ be states of systems G_{OE} and K_{OE}, $e = \tilde{x} - x$. Define*

$$\mathcal{H}_e(e,\tilde{x}) := \frac{\partial U}{\partial e}(e)(f(\tilde{x}) - f(x) + L_0(\tilde{x})(h_2(\tilde{x}) - h_2(x))) +$$

$$\frac{1}{4}\frac{\partial U}{\partial e}(e)(g_1(x) + L_0(\tilde{x})k_{12}(x))(g_1(x) + L_0(\tilde{x})k_{12}(x))^T \frac{\partial U^T}{\partial e}(e) +$$

$$(h_1^T(\tilde{x}) - h_1^T(x))(h_1(\tilde{x}) - h_1(x)) - \frac{\partial U}{\partial e}(e)(g_0(x) - g_0(\tilde{x}))h_1(\tilde{x})$$

with $L_0(\tilde{x})$ defined as in previous Theorem. Then for all $(x,\tilde{x}) \in \mathcal{B}_r$ with some $r > 0$, $\mathcal{H}_e(e,\tilde{x}) \le 0$. Moreover, there exists $\pi(e)$ (locally) positive definite such that $\mathcal{H}_e(e,\tilde{x}) + \pi(e) \le 0$.

Now, we prove theorem 3.6.

Proof

Consider $\Omega(G_{OE}, K_{FC})$ which has following realization

$$\left\{ \begin{array}{l} \dot{x} = f(x) - g_0(x)h_1(\tilde{x}) + g_1(x)w \\ \dot{\tilde{x}} = f(\tilde{x}) - g_0(\tilde{x})h_1(\tilde{x}) + L_0(\tilde{x})(h_2(\tilde{x}) - h_2(x) - k_{21}(x)w) \\ z = h_1(x) - h_1(\tilde{x}) \end{array} \right.$$

Let $e = \tilde{x} - x$, for $(x,\tilde{x}) \in \mathcal{B}_r$.

$$\dot{U}(e) = \frac{\partial U}{\partial e}(e)((f(\tilde{x}) - f(x) + L_0(\tilde{x})(h_2(\tilde{x}) - h_2(x)) -$$

$$(L_0(\tilde{x})k_{21}(x) + (g_1(x))w) - \frac{\partial U}{\partial e}(e)(g_0(\tilde{x}) - g_0(x))h_1(\tilde{x})$$

$$\le \|w\|^2 - \|z\|^2 - \|y_0\|^2 \le -\|z\|^2 + \|w\|^2 ;$$

the former inequality follows from the preceding lemma. Thus,

$$\int_0^T (\|w\|^2 - \|z\|^2)dt \ge U(e(T)) - U(0) = U(e(T)) \ge 0.$$

for all $T \ge 0$, which implies $\mathcal{L}_2$-gain ≤ 1. □

Theorem 3.8 *Under the assumption of the previous Theorem, if in addition, $L_1(x)$ is such that $\frac{\partial U}{\partial x}(x)L_1(x) = -2h_1^T(x)$, then the controller $u = \Omega(M_{OE}, Q)y$ with M_{OE} given by*

$$\left\{ \begin{array}{l} \dot{\tilde{x}} = f(\tilde{x}) - g_0(\tilde{x})h_1(\tilde{x}) + L_0(\tilde{x})h_2(\tilde{x} - y) + (g_2(\tilde{x}) + L_1(\tilde{x}))u_0 \\ u = -h_1(\tilde{x}) + u_0 \\ y_0 = h_2(\tilde{x}) - y \end{array} \right.$$

for all $Q \in \mathcal{FG}$ also makes the closed loop system (locally) has $\mathcal{L}_2$-gain ≤ 1.

Proof

Consider $\Omega(G_{OE}, \Omega(M_{OE}, Q))$ for $Q \in \mathcal{FG}$ which has following realization.

$$\left\{ \begin{array}{l} \dot{\xi} = a(\xi) + b(\xi)y_0 \\ u_0 = c(\xi) \end{array} \right.$$

And U_Q is a solution to the HJI with respect to Q with state ξ, then $\dot{U}_Q(\xi) \le \|y_0\|^2 - \|u_0\|^2$.

The similar argument shows that there exists $r > 0$, for $(x, \hat{x}, \xi) \in \mathcal{B}_r$,

$$\dot{U}(e) \le \|w\|^2 - \|z\|^2 - \|y_0\|^2 + \|u_0\|^2 - \pi(e)$$

for some locally positive definite $\pi(e)$. Thus,

$$\dot{U}(e) + \dot{U}_Q(\xi) \le -\|z\|^2 + \|w\|^2 - \pi(e) \le -\|z\|^2 + \|w\|^2$$

Therefore,

$$\int_0^T (\|z\|^2 - \|w\|^2)dt \le U(0) - U(e(T)) = -U(e(T)) \le 0.$$

for all $T \in \mathbf{R}^+$, which implies the $\mathcal{L}_2$-gain ≤ 1. □

4 Output Feedback $\mathcal{H}_\infty$-Control Problems

We now consider the OF $\mathcal{H}_\infty$ control problem. The solutions to this problem are based on the results in the last section.

4.1 Solutions to Output Feedback Problems

The nonlinear time-invariant plant is realized as control-affine state-space equation:

$$G : \left\{ \begin{array}{lllllll} \dot{x} & = & f(x) & + & g_1(x)w & + & g_2(x)u \\ z & = & h_1(x) & + & & & k_{12}(x)u \\ y & = & h_2(x) & + & k_{21}(x)w & & \end{array} \right.$$

where $f(0) = 0, h_1(0) = 0, h_2(0) = 0$; x, w, u, z, and y are assumed to have dimensions n, p_1, p_2, q_1, and q_2, respectively.

The following assumptions are made:

[A2]: $k_{12}^T(x) \begin{bmatrix} h_1(x) & k_{12}(x) \end{bmatrix} = \begin{bmatrix} 0 & I \end{bmatrix}$;

[A3]: $\begin{bmatrix} g_1(x) \\ k_{21}(x) \end{bmatrix} k_{21}^T(x) = \begin{bmatrix} 0 \\ I \end{bmatrix}$;

[A4]: $[h_1(x), f(x)]$ is zero-state detectable;

[A5]: $[h_2(x), f(x)]$ is zero-state detectable.

The main idea of construction is to convert the general problem OF into the simpler problems which have been solved.

Let $V(x) \geq 0$ be the solution of $\mathcal{H}_{FI}(V,x) \leq 0$. Define

$$F_0(x) := -\frac{1}{2}g_2^T(x)\frac{\partial V^T}{\partial x}(x), F_1(x) := \frac{1}{2}g_1^T(x)\frac{\partial V^T}{\partial x}(x),$$

and new variables $r := w - F_1(x)$ and $v := u - F_0(x)$. We get a new system

$$G_a : \left\{ \begin{array}{ccccccc} \dot{x} & = & f_a(x) & + & g_1(x)r & + & g_2(x)u \\ v & = & h_a(x) & + & & + & u \\ y & = & h_2(x) & + & k_{21}(x)r & & \end{array} \right.$$

where $f_a(x) := f(x) + g_1(x)F_1(x), h_a(x) := F_0(x)$.

Lemma 4.1 *Consider systems G and G_a. If the controller K makes $\Omega(G_a, K)$ have $\mathcal{L}_2$-gain ≤ 1, it also results in $\Omega(G,K)$ having $\mathcal{L}_2$-gain ≤ 1.*

This lemma can be proved by conducting the completion of squares [11]. Note that system G_a is of OE structure and satisfies the structure assumption **[A3]**.

Define

$$\mathcal{H}_a(W,x) := \frac{\partial W}{\partial x}(x)f_a(x) + \frac{1}{4}\frac{\partial W}{\partial x}(x)g_1(x)g_1^T(x)\frac{\partial W^T}{\partial x}(x) + $$
$$+h_a^T(x)h_a(x) - h_2^T(x)h_2(x).$$

Take $W(x) = U(x) - V(x)$ with $W(0) = U(0) - V(0)$ where $V(x) \geq 0$ is given just now. Note that

$$\mathcal{H}_a(W,x) = \mathcal{H}_{FC}(U,x) - \mathcal{H}_{FI}(V,x) = \mathcal{H}_{FC}(U,x) + \psi(x),$$

where $\psi(x) \geq 0$ is such that $\mathcal{H}_{FI}(V,x) + \psi(x) = 0$. Thus, $\mathcal{H}_a(W,x) \leq 0$ if and only if $\mathcal{H}_{FC}(U,x) + \psi(x) \leq 0$. Assume $U(x)$ is such that $\mathcal{H}_{FC}(U,x) + \psi(x) \leq 0$ has a positive definite Hessian matrix at $x = 0$, then $\mathcal{H}_a(W,x)$ also has negative definite Hessian matrix at 0. Suppose $L_0(x)$ is such that $\frac{\partial W}{\partial x}(x)L_0(x) = -2h_2^T(x)$. The controller K for the new OE structure given by Theorem 3.6 is

$$K : \left\{ \begin{array}{l} \dot{\hat{x}} = f_a(\hat{x}) + g_2(\hat{x})h_a(\hat{x}) + L_0(\hat{x})h_2(\hat{x}) - L(\hat{x})y \\ u = h_a(\hat{x}) \end{array} \right.$$

is such that system $\Omega(G_a, K)$ locally has $\mathcal{L}_2$-gain ≤ 1.

By lemma 4.1, $\Omega(G,K)$ has $\mathcal{L}_2$-gain ≤ 1. Next, we examine the stability of the closed loop system $\Omega(G,K)$ which has the following realization,

$$\left\{ \begin{array}{l} \dot{x} = f(x) + g_2(x)F_0(\hat{x}) + g_1(x)w \\ \dot{\hat{x}} = f_K(\hat{x}) + L_0(\hat{x})(h_2(\hat{x}) - h_2(x)) + L_0(\hat{x})k_{21}(x)w \\ z = h_1(x) + k_{12}(x)F_0(\hat{x}) \end{array} \right.$$

where

$$f_K(\hat{x}) := f(\hat{x}) + g_1(\hat{x})F_1(\hat{x}) + g_2(\hat{x})F_0(\hat{x}). \quad (4.1)$$

Take $e = \hat{x} - x$. Note that $\mathcal{H}_a(W, \cdot)$ has negative definite Hessian matrix as does $\mathcal{H}_{FC}(U, \cdot)$. Using the same technique as in the proof of Theorem 3.4, it can be concluded that for some locally positive definite $\pi : \mathbf{R}^n \rightarrow \mathbf{R}^+$, such that if $(x, \hat{x}) \in \mathcal{B}_s$ for some $s > 0$

$$\dot{W}(e) \leq ||r||^2 - ||v||^2 - \pi(e)$$

Let $L_{OF}(x,e) = V(x) + W(e)$ with $e = \hat{x} - x$. By assumption $V(x)$ and $W(e)$ are positive definite, so is $L(x,e)$, and it can be used as a Lyapunov function. Take $w = 0$,

$$\dot{V}(x) \leq -||z||^2 + ||v||^2 - ||r||^2$$
$$\dot{L}_{OF}(x,e) = \dot{V}(x) + \dot{W}(e) \leq -||z||^2 - \pi(e) \leq 0.$$

Then $\dot{L}_{OF}(x,e) = 0 \Rightarrow z = 0$ and $\pi(e) = 0 \Rightarrow x = 0$ and $e = 0$. Therefore, $\dot{L}_{OF}(x,e)$ is locally negative definite, the closed loop system is locally asymptotically stable.

Therefore, we have the following results about output $\mathcal{H}_\infty$-control problem,

Theorem 4.2 *Consider G, if there is some $\psi(x) \geq 0$ with $\psi(0) = 0$ such that*

(i) there exists a positive definite $V(x)$ which solves the HJE: $\mathcal{H}_{FI}(V,x) + \psi(x) = 0$ with $V(0) = 0$.

(ii) there exists a positive definite $U(x)$ which satisfies the HJI: $\mathcal{H}_{FC}(U,x) + \psi(x) \leq 0$ with $U(0) = 0$. And $\mathcal{H}_{FC}(U,x) + \psi(x)$ has nonsingular Hessian matrix at 0.

(iii) $U(x) - V(x) \geq 0$ is positive definite. And

$$(\frac{\partial U}{\partial x}(x) - \frac{\partial V}{\partial x}(x))L_0(x) = -2h_2^T(x),$$

has a solution $L_0(x)$. Then the $\mathcal{H}_\infty$-control problem is (locally) solvable. Furthermore,

$$K : \left\{ \begin{array}{l} \dot{\hat{x}} = f_K(\hat{x}) + L_0(\hat{x})h_2(\hat{x}) - L_0(\hat{x})y \\ u = F_0(\hat{x}) \end{array} \right.$$

is such a controller.

Note that $\mathcal{H}_\infty$-controllers have separation structures. The separation principle for the $\mathcal{H}_\infty$-performance in nonlinear systems was first confirmed by Ball-Helton-Walker[1] (see also Isidori [6]). Similar arguments to Theorems 3.8 and 4.2 can be also used to examine the controller parametrization.

Theorem 4.3 *Consider a system G satisfying the condition in Theorem 5.1. If in addition $L_1(x)$ satisfies*

$$(\frac{\partial U}{\partial x}(x) - \frac{\partial V}{\partial x}(x))L_1(x) = -2h_1^T(x),$$

then the controller $u = \Omega(M,Q)y$ with M given by

$$\left\{ \begin{array}{ccccccc} \dot{\tilde{x}} & = & f_K(\tilde{x}) & - & L_0(\tilde{x})y & + & (g_2(\tilde{x}) + L_1(\tilde{x}))u_0 \\ u & = & F_0(\tilde{x}) & + & & & u_0 \\ y_0 & = & h_2(\tilde{x}) & - & y & & \end{array} \right.$$

for all $Q \in \mathcal{FG}$ also (locally) solves OF $\mathcal{H}_\infty$-control problem.

Proof

By lemma 4.1 and theorem 3.8 it follows that the closed loop system $\Omega(G,K)$ with $K = \Omega(M,Q)$ has the $\mathcal{L}_2$-gain ≤ 1. Now it is sufficient to consider the stability issue. Suppose Q has the following realization

$$\left\{ \begin{array}{l} \dot{\xi} = a(\xi) + b(\xi)y_0 \\ u_0 = c(\xi) \end{array} \right.$$

and $U_Q(\xi)$ is such that $\dot{U}_Q(\xi) \leq ||u_0||^2 - ||y_0||^2$. Take $w = 0$, the closed loop system has following hierarchical structure,

$$\left\{ \begin{array}{l} \dot{x} = f(x) + g_2(x)(F_0(\tilde{x}) + c(\xi)) \\ \dot{\xi} = a(\xi) + b(\xi)(h_2(\tilde{x}) - h_2(x)) \\ \dot{\tilde{x}} = f_K(\tilde{x}) + L_0(\tilde{x})(h_2(\tilde{x}) - h_2(x)) + (g_2(\tilde{x}) + L_1(\tilde{x}))c(\xi) \end{array} \right.$$

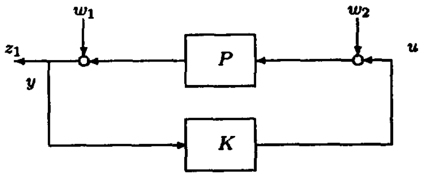
Let $V, W : \mathbf{R}^n \rightarrow \mathbf{R}^+$ positive definite be defined as in the preceding discussion. Denote $e = \tilde{x} - x$. Similar arguments to theorems 3.8 and 4.2 show that

$$\dot{W}(e) \leq ||r||^2 - ||v||^2 - ||y_0||^2 + ||u_0|| - \pi(e)$$

for some positive definite $\pi : \mathbf{R}^n \rightarrow \mathbf{R}^+$. Define $L_{OF}(x,e,\xi) := V(x) + W(e) + U_Q(\xi)$ as the Lyapunov function of the closed loop system, then $\dot{L}_{OF}(x,e,\xi) \leq -||z||^2 - \pi(e)$. Now $\dot{L}_{OF}(e,\xi) = 0 \Rightarrow \pi(e) = 0$ and $||z|| = 0$, so $e = 0$ and $z = 0$; the latter implies $x(t) \rightarrow 0$ as $t \rightarrow \infty$ by **[A4]**; on the other hand if $e = 0, x = 0$, then $\dot{\xi} = a(\xi)$, which is asymptotically stable and $\xi(t) \rightarrow 0$ as $t \rightarrow \infty$. The interconnected (e, ξ) is locally asymptotically stable by LaSalle's theorem and Vidyasagar's theorem [17]. □

4.2 Examples

This example is basically taken from [3]. The block diagram is as follows



Where P is the nonlinear plant; K is the controller to be designed such that the output z_1 is regulated; y is the measured output, based on which the control action u is produced; w_2 is the disturbance from the actuator; and w_1 is the noise from the sensor. The **control problem** is to design the controller K such that the influence of the noises w_1 and w_2 on the regulated output z_1 can be reduced to the minimal with the reasonable effort (control action should not be too large). Let $r \geq 0$, the $\mathcal{H}_\infty$-control problem in this setting can be formulated as: Given $\gamma > 0$, find a controller K such that

$$\int_0^T (\|z_1\|^2 + r\|u\|^2)dt \leq \gamma^2 \int_0^T (\|w_1\|^2 + \|w_2\|^2)dt, \forall T \in \mathbf{R}^+$$

In this example, the plant has the following realization:

$$\begin{cases} \dot{x} = e^x(w_2 + u) \\ z_1 = x + w_1 \\ y = x + w_1 \end{cases}$$

We will consider two cases. In both cases, since the stability of the resulting closed loop systems can be easily checked by using the corresponding Theorems, we just consider the $\mathcal{H}_\infty$-performances.

Case I: $r = 0$

Consider the control problem that a controller K is designed such that:

$$\int_0^T \|z_1\|^2\, dt \leq \gamma_0^2 \int_0^T (\|w_1\|^2 + \|w_2\|^2)dt, \forall T \in \mathbf{R}^+$$

where $\gamma_0 = 1/(1-\epsilon)$ for some $0 < \epsilon < 1$.

Take $w := \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ and $z := (1-\epsilon)z_1$, the system can be transformed into a new system as follows (see [14]).

$$\begin{cases} \dot{x} &= & & \begin{bmatrix} 0 & e^x \end{bmatrix} w & + & u_N \\ z &= & \frac{1-\epsilon}{\sqrt{2\epsilon-\epsilon^2}}x & & & \\ y_N &= & \frac{1}{\sqrt{2\epsilon-\epsilon^2}}x & + \begin{bmatrix} 1 & 0 \end{bmatrix} w & & \end{cases}$$

with $u_N := e^x u$ and $y_N = \sqrt{2\epsilon - \epsilon^2}y$. Now the system has a output-injection structure.

Consider the HJI with respect to this structure:

$$\frac{\partial U}{\partial x}(x) \cdot 0 + \frac{1}{4}e^{2x}(\frac{\partial U}{\partial x}(x))^2 + (\frac{1-\epsilon}{\sqrt{2\epsilon-\epsilon^2}}x)^2 - (\frac{1}{\sqrt{2\epsilon-\epsilon^2}}x)^2 \leq 0$$

A class of positive solutions $U(x)$ satisfy $\frac{\partial U}{\partial x}(x) = 2\rho e^{-x}x$ for $0 \leq \rho \leq 1$. Take $\rho = 1$, then $L(x) = -\frac{e^x}{\sqrt{2\epsilon-\epsilon^2}}$ satisfies

$$\frac{\partial U}{\partial x}(x)L(x) = \frac{2x}{\sqrt{2\epsilon - \epsilon^2}}.$$

It follows that the controller is

$$u_N = L(x)y_N = -\frac{e^x}{\sqrt{2\epsilon - \epsilon^2}} \cdot \sqrt{2\epsilon - \epsilon^2}y = -e^x y$$

or the output-injection can be recovered as $u = -y$. Note that it is independent of ϵ.

This $\mathcal{H}_\infty$ controller is identity ($K = -1$). Actually, we have following general result which is proved in [3].

Theorem 4.4 *Consider the feedback system as shown. Suppose the plant P with the same dimensional input and output vectors is passive, and $K = -I$, then*

$$\int_0^T \|z_1\|^2\, dt \leq \int_0^T (\|w_1\|^2 + \|w_2\|^2)dt, \forall T \in R^+$$

Case II: $r = 1$

Consider the control problem that a controller K is designed such that:

$$\int_0^T (\|z_1\|^2 + \|u\|)dt \leq \gamma_0^2 \int_0^T (\|w_1\|^2 + \|w_2\|^2)dt, \forall T \in \mathbf{R}^+.$$

Using the similar manipulation in Case I, it can be verified by theorem 4.2 that if $\gamma_0 = \sqrt{2}/(1-\epsilon)$ for all $0 < \epsilon < 1$ then the output feedback $\mathcal{H}_\infty$-control problem have solution. So the optimal $\gamma_0 \leq \sqrt{2}$.

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